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Derivation of Critical Exponents from Extreme Value Distributions

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It is shown that all critical exponents, with the exception of the heat capacity exponent other than for the mean field theory, can be derived from the characteristic exponent of an extreme value distribution for the smallest value and the dimensionality of the space. The relation between the characteristic exponent and the dimensionality d of the space imposes the condition $d \leq 4$. This is borne out by direct evaluation of the spatial correlation function.

1. EXTREME VALUE DISTRIBUTIONS

In contrast to the usual thermodynamic situation, where there is an overwhelming tendency to cluster about mean values as the number of random variables increases without limit, other types of phenomena tend to be governed by their distribution of extreme values. Well-known examples are natural disasters (e.g., floods, which follow double exponential distributions for largest values), the breaking strength of materials (described by the Weibull distribution for smallest values), the distribution of incomes and word frequencies (inverse power laws known as the Pareto-Zipf distribution), and the statistics of evolutionary processes. All these phenomena are governed by what Lévy termed stable laws, which, while possessing a domain of attraction like that of the normal distribution, do not possess finite second moments. Stable laws have had relatively little impact in the physical sciences, apart from the appearance of the Cauchy distribution in the study of spectral line shapes, where it is known as the Lorentz distribution. The aim of this paper is to call attention to the fact that stable laws are responsible for the scaling laws in the study of critical phenomena.

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We will be interested in the minimum value

$$\hat{X}_n = \min(X_1, X_2, \ldots, X_n)$$

of a set of n independent and identically distributed observations or measurements. This is because we are, in effect, appealing to the principle that the "weakest link determines the strength of the chain." Because $\hat{X}_n \geq x$ means that each $X_j \geq x$ and because they are independent and identically distributed random variables with a common distribution function $F(x)$, it follows that

$$\begin{aligned} 1 - \Pr\{\hat{X}_n > x\} &= 1 - [1 - F(x)]^n \\ &= 1 - \left[1 - \frac{nF(x)}{n}\right]^n \\ &\to 1 - e^{-nF(x)} \end{aligned} \tag{1}$$

Since for asymptotically large values of n the extremes increase, in absolute value, without bound, we know that there cannot be any asymptotic distribution without normalization, just as in the case of the central limit theorem. This is guaranteed by the property of stability for which it is necessary and sufficient that the distribution defined by $nF(x)$ belongs to the same family as that defined by $F(x)$. This requires that it differ by a positive scale factor $\lambda(n)$ such that $nF(x) = F(\lambda(n)x)$. Now, if we consider two integers n_1 and n_2, it follows that (de Finetti, 1975, p. 99)

$$n_1 n_2 F(x) = F(\lambda(n_1)\lambda(n_2)x) = F(\lambda(n_1 n_2)x)$$

from which it follows that $\lambda(n_1)\lambda(n_2) = \lambda(n_1 n_2)$. This functional relation is characteristic of powers, so we have $\lambda(n) = n^{1/k}$, where k is referred to as the characteristic exponent. Therefore, we have

$$nF(x) = F(n^{1/k}x) = F(m) = \left(\frac{m}{m_0}\right)^k \tag{2}$$

where m_0 is a constant scale factor, independent of n.[2] The expression for the initial distribution (2) can be looked upon as the ratio of the volume of a hypersphere of k dimensions occupied by the system, m^k, to that of the total volume of a hypersphere of n dimensions, m_0^k, available to the system.

[2]This derivation of the Weibull distribution differs from the usual derivation found in most texts (see, for example, Galambos, 1978, pp. 189–191), which begins with the distribution for the largest value and uses the symmetry between the distributions of the largest and smallest values. Rather, it is more in line with the usual derivation of the microcanonical ensemble whose unnormalized distribution is proportional to the volume of phase space occupied by the system (Khinchin, 1949, pp. 29ff.).

Hence, introducing (2) into (1), we obtain

$$\Pr\{\hat{M} \leq m\} = 1 - e^{-(m/m_0)^k} \tag{3}$$

which is known in engineering circles as the Weibull distribution, after the Swedish engineer who used it for the first time in the analysis of the breaking strength of materials (Gumbel, 1958, p. 279).

2. THERMODYNAMICS OF RARE EVENTS

The connection with thermodynamics is obtained by noting that the tail of the asymptotic distribution (3) can be derived from the relation

$$\Pr(\hat{M} > m) = \left[1 - \frac{1}{n}\left(\frac{m}{m_0}\right)^k\right]^n \tag{4}$$

in the limit as $n \to \infty$. According to the generalized Boltzmann relation for asymptotic distributions of smallest values, the entropy reduction due to a finite value of m, which behaves as an order parameter, is (Lavenda and Florio, 1992)

$$\Delta S(m) \equiv S(m) - S_0 = \ln \Pr(\hat{M} > m) = n \ln\left[1 - \frac{1}{n}\left(\frac{m}{m_0}\right)^k\right] \tag{5}$$

in units where Boltzmann's constant is unity, where S_0 is the entropy of the reference state $m = 0$. This expression is comparable with the asymptotic form of the expression for the "entropy of mixing" for all probability distributions in the domain of attraction of the normal law.

Differentiating (5), we get

$$S'(\sigma) = -\frac{k\sigma^{k-1}}{1 - n^{-1}\sigma^k} \tag{6}$$

where we introduced the reduced variable $\sigma \equiv m/m_0$. But, by the second law, this must be equal to

$$S'(\sigma) = \left(\frac{\partial S}{\partial \sigma}\right)_E + \left(\frac{\partial S}{\partial E}\right)_\sigma E'(\sigma) \tag{7}$$

where E is the internal energy. Equating the two expressions for the entropy derivative (6) and (7), we find

$$h = E' + \frac{k\sigma^{k-1}T}{1 - n^{-1}\sigma^k} \tag{8}$$

where $h = -T(\partial S/\partial \sigma)_E$ is the reduced magnetic field.

In order to determine the form of the energy difference ΔE we observe that the change in the Helmholtz potential is

$$\Delta A(\sigma, T) = \Delta E - T\,\Delta S = \Delta E - nT\ln\left(1 - \frac{\sigma^k}{n}\right) \tag{9}$$

It is easily seen that $(\partial A/\partial\sigma)_T = h$ is equivalent to (8), which determines a nonvanishing value of the order parameter.

The bifurcation point separating zero from nonzero values of the magnetization is determined by the vanishing of the kth-order derivative of the Helmholtz potential, viz.,

$$A^{(k)}(0, T_c) = E^{(k)} + k!\,T_c = 0 \tag{10}$$

where T_c is the critical temperature. This implies that the energy difference must have the form

$$\Delta E = -T_c\sigma^k \tag{11}$$

implying that ΔE must necessarily be a *concave* function of σ (Lavenda, 1991, p. 324).[3] Hence, *only for extreme value distributions can a phase transition of order k occur in which $k > 2$.*

Another, yet equivalent way of looking at (7) with E' given by the derivative of (11) is to consider that the total field, equal to the thermodynamic force,

$$\chi = h - E'(\sigma) \tag{12}$$

given by the sum of an external magnetic field h and an internal field $E' = -\lambda k\sigma^{k-1}$ created by the interactions among the spins, where λ is the molecular field parameter, which turns out to be proportional to the critical temperature (Stanley, 1971). For $k = 2$, it reduces to the mean field approximation governed by the normal law, while for $k > 2$ there is an enhancement of the molecular interactions which are governed by extreme value distributions.

[3]Herein lies the difference from phase transitions that are derived from entropies of mixing belonging to either the negative binomial or binomial distribution. According to the central limit theorem, these distributions merge into the Gaussian distribution in the limit, so that a Taylor series starts with quadratic terms and hence the order of the derivative in (10) is $k = 2$. This corresponds to mean field theory. But because we are dealing with extreme value distributions for the smallest value, the smallest order terms in the series expansion of the logarithm will start with terms that are of order $k > 2$. Therefore, a qualification in Lavenda (1991, p. 324) must be made in which there are no higher than second-order phase transitions in any system that is governed by a probability distribution which is attracted to the normal law.

In the absence of an external field, (8) has a nontrivial solution:

$$\sigma_0 = (n|\varepsilon|)^{1/k} \tag{13}$$

for $T < \lambda$, where the dimensionless temperature variable $\varepsilon = (T - T_c)/T_c$. In contrast with the mean field theory, where the square of the zero-field magnetization vanishes linearly with the temperature difference $|\varepsilon|$, equation (13) predicts that the magnetization raised to the power k will vanish in the same manner.

We have shown (Lavenda and Florio, 1992) that this characteristic exponent is related to the characteristic exponent of stable laws ϑ in the interval $1 < \vartheta \leq 2$ according to

$$k = \vartheta/(\vartheta - 1)$$

implying that $2 \leq k < \infty$. The limit where the normal distribution is attained is $\vartheta = 2$, which is included, while the Cauchy limit, corresponding to $\vartheta = 1$, requires a separate analysis. We shall now show that all the critical exponents can be derived from the characteristic exponent k of the Weibull distribution (3).

3. CRITICAL EXPONENTS

3.1. Critical Isotherm Exponent δ

The critical isotherm exponent δ is defined at the critical temperature as $h \sim \sigma^{\delta}$, where $\sim$ means singular part of. From (8) we get $h \sim \sigma^{2k-1}$ identifying δ as

$$\delta = 2k - 1 = \frac{\vartheta + 1}{\vartheta - 1}$$

and therefore

$$3 \leq \delta < \infty$$

3.2. Magnetization Exponent β

The zero-field magnetization introduces the critical exponent β as $\sigma \sim |\varepsilon|^{\beta}$. Setting $h = 0$ in (8), we get $\sigma \sim |\varepsilon|^{1/k}$ with the consequence that

$$\beta = 1/k = (\vartheta - 1)/\vartheta$$

This implies that the range of β is

$$0 < \beta \leq 1/2$$

3.3. Susceptibility Exponent γ

The inverse isothermal susceptibility is defined as $\chi_T^{-1} = (\partial h/\partial \sigma)_T$. Below the critical temperature, it scales as $\chi_T^{-1} \sim |\varepsilon|^{\gamma}$. From (8) we find the inverse isothermal susceptibility, evaluated at $h=0$, as

$$\chi_T^{-1} = n^{(k-2)/k} \frac{(kT_c)^2}{T} |\varepsilon|^{2(k-1)/k} \tag{14}$$

implying that

$$\gamma = 2(k-1)/k = 2/\vartheta$$

Hence, the range of values for this critical exponent is

$$1 \leq \gamma < 2$$

3.4. Specific Heat Exponent α

The specific heat exponent α, defined by $C_h \sim |\varepsilon|^{-\alpha}$, can most easily be obtained by introducing (13) into the entropy reduction (5) to obtain

$$\Delta S = n \ln\left(\frac{T}{T_c}\right) \tag{15}$$

which has the same form as the entropy difference of an ideal gas when n is identified as half the number of degrees of freedom. Below the critical point, the entropy change is always negative. Differentiating with respect to T yields

$$C_h = T\left(\frac{\partial S}{\partial T}\right)_h = n \tag{16}$$

and consequently

$$\alpha = 0 \tag{17}$$

which is also a mean field result.

How this comes about can be seen from the relation

$$C_h - C_m = T\chi_T^{-1}\left(\frac{\partial \sigma}{\partial T}\right)_h^2$$

From (9) and the definition of the heat capacity at constant magnetization, $C_m = -T(\partial^2 A/\partial T^2)_m = 0$, for all T. This is the same as the mean field theory. At zero field we find from (13)

$$T\left(\frac{\partial \sigma}{\partial T}\right)_h = -\frac{\beta n^{\beta}}{T_c} |\varepsilon|^{-(1-\beta)}$$

the square of whose exponent is the negative of γ. Hence, we obtain (17) and (16).

3.5. Spatial Correlation Exponents ν and η

So far we have only introduced critical exponents that are related to purely thermodynamic quantities. However, there are two other critical exponents which are related to spatial correlations. The first is defined by

$$\xi \sim |\varepsilon|^{-\nu}$$

where ξ is the correlation length. The second is defined in terms of the correlation function $G(r)$ at the critical point:

$$G_c(r) \sim r^{-(d-2+\eta)} \tag{18}$$

where d is the dimensionality of the system. This exponent was introduced by Fisher (1967) to take into account the slight concavity of the curves in the plot of the inverse scattering intensity versus the square of the modulus of the momentum transfer vector. Since extreme value distributions describe ideal behavior, we must set $\eta \equiv 0$ as in the classical theory. The usual scaling argument that $G_c \sim |\varepsilon|^{\nu(d-2)}$, which has the same dimension as the variance $\overline{(\Delta\sigma)^2} = T\chi_T/V \sim \xi^{-d}|\varepsilon|^{-\gamma} \sim |\varepsilon|^{\nu d-\gamma}$ in a volume of size $V \sim \xi^d$,[4] yields the classical result $2\nu = \gamma$, and hence

$$\nu = (k-1)/k = 1/\vartheta \tag{19}$$

Expression (19) identifies the critical exponent ν with the inverse of the characteristic exponent and with a range

$$\tfrac{1}{2} \leq \nu < 1$$

Since we also have $\overline{(\Delta\sigma)^2} \sim |\varepsilon|^{2\beta}$, we obtain $\nu d - \gamma = 2\beta$, which is a combination of the Rushbrooke and Josephson scaling laws. This identifies the dimensionality as twice the characteristic exponent:

$$d = 2k/(k-1) = 2\vartheta \tag{20}$$

The classical predictions $\gamma = 1$ and $\nu = \frac{1}{2}$ imply that $k = 2$, which identifies the extreme value distribution as the Rayleigh distribution, and from (20), it fixes the dimensionality $d = 4$. This is its maximum value since $1 < \vartheta \leq 2$.

We recall that if two random variables are independent and normally distributed, then their length is a random variable having a Rayleigh density.

[4]We are assuming, as is commonly done, that the correlation length, or some multiple of it, determines a "sufficiently" large volume. This is the weak link in the argument that makes the results (19) and (20) less general than those previously obtained.

The scale factor is now the standard deviation, which varies as the square root of n, the number of independent and identically distributed random variables. Since the mean value is proportional to n, the relative fluctuation, which is the ratio of the mean value to the standard deviation, decays as $1/\sqrt{n}$, as n increases without bound and we obtain the usual thermodynamic situation where there is a tendency to cluster about the mean value in this limit. This is not true for stable distributions with $\vartheta < 2$, since they have infinite variance.

In general, the mean field exponents do not satisfy Josephson's law, since they are independent of the dimensionality of the space (Huang, 1987, p. 425), so (20) is not a necessary consequence because Josephson's law has gone into its derivation. Nevertheless, for $k=2$, the Rayleigh distribution results and it is this distribution which governs the fluctuations about the mean field quantities. Moreover, if $d=4$, then we get the "ideal limit," exactly as in the case of a polymer chain in 4 dimensions (de Gennes, 1979). For an ideal chain in 4 dimensions, the maximum repulsive energy becomes independent of the number of monomer units, so that excluded-volume effects become negligible. In analogy to the ratio of the maximum repulsive energy to the elastic energy of a polymer chain, we take the ratio

$$G_c(\xi)/\overline{(\Delta\sigma)^2} \sim \xi^{-(d-2)}/\xi^{-2}$$

of the correlation function at the critical point, (18), to the variance of the limiting Rayleigh distribution, $\overline{(\Delta\sigma)^2} \sim \xi^{-2}$, as a measure of the strength of the fluctuations. For $d=4$, the ratio is independent of the correlation length and there is no divergence in the limit as $\xi \to \infty$. The system behaves ideally, even in the critical region.

To prove that the maximum dimension is, indeed, $d_{\max}=4$, we consider the ratio of the observed scattering intensity $I(q)$ to the ideal, low-density scattering intensity $I_0(q)$, where q is the magnitude of the momentum transfer vector; this ratio is usually assumed to be given by the Lorentzian form (Fisher, 1967)

$$S(q) = \frac{I(q)}{I_0(q)} \propto \frac{1}{\kappa^2 + q^2} \tag{21}$$

at least away from the critical point. This expression defines the structure factor $S(q)$, where $\kappa = \xi^{-1}$. The Fourier transform of (21) in d dimensions gives the correlation function $G(r)$ (Stanley, 1971)

$$G(r) = \int_0^\infty S(q) \int_0^\pi e^{iqr\cos\varphi} \sin^{d-2}\varphi \, d\varphi \int d\Omega_{d-1}\, q^{d-1}\, dq$$

where $d\Omega_{d-1}$ is an element of solid angle in $d-1$ dimensions. Apart from the numerical factors, the correlation function is given by the expression

$$G(r) \propto \int_0^\infty S(q) \frac{J_{d/2-1}(qr)}{(qr)^{d/2-1}} q^{d-1}\, dq$$

$$= \frac{1}{r^{d-2}} \int_0^\infty x^{d/2} J_{d/2-1}(x) \frac{dx}{(\kappa r)^2 + x^2}$$

$$= \frac{1}{r^{d-2}} (\kappa r)^{d/2-1} K_{d/2-1}(\kappa r) \tag{22}$$

provided $d < 5$ (Gradshteyn and Ryzhik, 1980, formula 6.566-2), where J_l is a Bessel function of the first kind and K_l is a Basset function. Although the result is well known, its limitation to dimensions less than 5 has gone unnoticed, or at least it has not been sufficiently emphasized because authors refer to d dimensions in a completely arbitrary fashion (de Gennes, 1979). As $r \to \infty$, we obtain

$$G(r) \propto \frac{\kappa^{(d-3)/2}}{r^{(d-1)/2}} e^{-\kappa r} \left\{ 1 + \frac{(d/2)^2 - d + \frac{3}{4}}{2\kappa r} + \cdots \right\} \tag{23}$$

It is quite remarkable that for the special case $d = 3$, all correction terms cancel and we get precisely the Ornstein–Zernike result

$$G(r) \propto \frac{e^{-\kappa r}}{r}$$

However, it should be observed that (23) is valid provided $d \leq 4$.

4. SCALING RELATIONS AND NUMERICAL VALUES

Apart from α and η, which in some way relate to deviations from extreme value distributions, the scaling laws for the critical exponents result in mere identities since each critical exponent can be expressed in terms of a single parameter—the characteristic exponent of the extreme value distribution. For instance, the Widom scaling law $\gamma = \beta(\delta - 1)$ is none other than the identity $2/\vartheta = 2/\vartheta$. The Rushbrooke equality $\alpha + 2\beta + \gamma = 2$ simply reduces to $2/k + 2(k-1)/k = 2$, which is a consequence of the magnetization law in zero field, (13), and the expression for the isothermal susceptibility, (14). These expressions cannot be tampered with without modifying the entire structure of the definitions of the critical point and the scaling

Table I. Comparison of Our Results Based on the Value of the Characteristic Exponent and the Critical Exponents for Various Models[a]

System	$\delta(T=T_c)$	k	β	γ	α	ν
Experimental range	4–5	—	0.32–0.39	1.3–1.4	0–0.14	0.6–0.7
Predicted range	3–∞	—	0–0.5	1–2	0	0.5–1
Classical	3	2	$\frac{1}{2}$	1	0 (*disc*)	$\frac{1}{2}$
Ising, $d=2$	15	8	$\frac{1}{8}$	$\frac{7}{4}$	0 (*log*)	$\frac{7}{8}$(1)
Ising, $d=3$	5	3	$\frac{1}{3}$ (0.31)	$\frac{4}{3}$(1.25)	0 (0.12)	$\frac{2}{3}$ (0.64)
Heisenberg, $d=3$	—	3	$\frac{1}{3}$ (0.3)	$\frac{4}{3}$(1.4)	0 (−0.14)	$\frac{2}{3}$ (0.7)

[a]Experimental range indicates experimental values taken from a variety of systems after Patashinskii and Pokrovskii (1973, Table 3, pp. 42–43). Predicted range is the range we have predicted from the characteristic exponent of the extreme value distribution.

exponents. Hence, systems with a nonvanishing α, or η, are nonideal in which deviations from extreme value distributions should be observed.

A comparison of the results with several known model systems is shown in Table I for $T<T_c$. Where differences or qualifications occur, they are shown in parentheses.

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